Quantum Analogs
Chapter 2

Student Manual

Modeling a Hydrogen Atom
with a
Spherical Resonator

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2. Modeling a hydrogen atom with a spherical resonator

Background:

The hydrogen atom, with a single electron in the Coulomb potential of the nucleus, is an ideal object for studying the basic principles of atomic physics. As the simplest of all atoms, without any electron correlations, it can be solved analytically.

The spherical symmetry of the three-dimensional problem makes it possible to separate the angular and radial variables for the solution of Schrödinger's equation. The acoustic analog uses a spherical resonator that allows a separation of variables for the solution of the Helmholtz equation in the same way as is done for the hydrogen atom. We will see that the eigenfunctions with respect to the angular variables – the spherical harmonics $Y_l^m(\theta,\varphi)$ – are exactly the same for both problems. The radial eigenfunctions, however, are different.

The three-dimensional Schrödinger equation

$$E\psi(\vec{r}) = -\frac{\hbar^2}{2m}\Delta\psi(\vec{r}) - \frac{e^2}{r}\psi(\vec{r}) \qquad (2.1)$$

expressed in polar coordinates

$$E\psi = \frac{\hbar^2}{2mr^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{\hbar^2}{2mr^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\hbar^2}{2mr^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} - \frac{e^2}{r}\psi$$

can be separated in two differential equations with the ansatz

$$\psi(r,\theta,\varphi) = Y_l^m(\theta,\varphi)\,\chi_l(r)\,. \qquad (2.2)$$

The spherical harmonics are solutions of the differential equation

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_l^m(\theta,\varphi) = l(l+1)Y_l^m(\theta,\varphi) \qquad (2.3)$$

and $\chi_l(r)$ is a solution of the so called radial equation

$$-\frac{\hbar^2}{2mr}\frac{\partial^2}{\partial r^2}r\chi(r) - \frac{l(l+1)\hbar^2}{2mr^2}\chi(r) - \frac{e^2}{r}\chi(r) = E\chi(r)\,. \qquad (2.4)$$

In the case of the spherical acoustic resonator we transform eqn. 1.4

$$\frac{\partial^2 p}{\partial t^2} = \frac{1}{\rho\kappa}\Delta p \qquad (2.5)$$

with the ansatz $p(\vec{r},t) = p(\vec{r})\cos(\omega t)$ into the time independent Helmholtz equation

$$\omega^2 p(\vec{r}) = -\frac{1}{\rho\kappa}\Delta p(\vec{r})\,, \qquad (2.6)$$

Using c as the speed of sound, equation 2.6 can be written as

$$-\frac{\omega^2}{c^2}p(\vec{r}) = \Delta p(\vec{r}) \qquad (2.7)$$

The Helmholtz equation in polar coordinates is given by

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right)-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p}{\partial\theta}\right)-\frac{1}{r^2\sin^2\theta}\frac{\partial^2 p}{\partial\varphi^2}=\frac{\omega^2}{c^2}p$$

It separates into a radial-function *f(r)* and the spherical harmonics $Y_l^m(\theta,\varphi)$.

$$p(r,\theta,\varphi)=Y_l^m(\theta,\varphi)\ f(r) \tag{2.8}$$

With this ansatz the Helmholtz equation is separated in one differential equation for the spherical harmonics

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_l^m(\theta,\varphi)=l(l+1)\ Y_l^m(\theta,\varphi) \tag{2.9}$$

and another for the radial function

$$-\frac{\partial^2 f}{\partial r^2}-\frac{2}{r}\frac{\partial f}{\partial r}+\frac{l(l+1)}{r^2}f(r)=\frac{\omega^2}{c^2}f(r) \tag{2.10}$$

You see immediately that eqn. 2.3 and eqn. 2.9 are exactly the same and have the same eigenfunctions and eigenvalues for the quantum numbers *l* (angular momentum or azimuthal quantum number) and *m* (magnetic quantum number). The radial equations are different, which, of course, results in different solutions. The Coulomb potential only appears in the radial equation (eqn. 2.4). Therefore, it does not affect the spherical harmonics. The eigenvalues of the radial equations are numerated by the quantum number n' (radial quantum number).

The energy levels $E_{n'l}$ of the hydrogen atom are the eigenvalues of the radial equation (2.4) and the eigenfrequencies of the spherical acoustic resonator $\omega_{n'l}$ are eigenvalues of the radial equation (2.10). Since the two differential equations are of different form, the resonance frequencies in the resonator can not be compared quantitatively with the energy levels of the hydrogen atom. However, the resonances can be classified with the same quantum numbers n'(radial quantum number), *l* (azimuthal quantum number) and *m* (magnetic quantum number). The quantum numbers are integers and

$$n'\geq 0 \qquad l\geq 0 \qquad -l\leq m\leq l \tag{2.11}$$

In the non-relativistic description of the hydrogen atom, many energy levels are degenerate, due to the special form of the Coulomb potential. The energies can be written in the form

$$E_{n'l}=-\left(\frac{e^2}{\hbar c}\right)^2\frac{mc^2}{2(l+1+n')^2}\,. \tag{2.12}$$

All levels with the same value for $(l+1+n')$ are degenerate. Therefore, a new quantum number is introduced that is called the "principal quantum number" *n*. It is given by

$$n=l+1+n' \tag{2.13}$$

For a given principal quantum number *n* the azimuthal quantum number *l* can take the values

$$0\leq l\leq n-1 \tag{2.14}$$

even though it runs to infinity for a given radial quantum number.

In the diagrams of the hydrogen atom spectrum shown below, the energy levels are labeled in two different ways. In the left figure they are labeled in the ordinary manner, using the principal quantum number. The right figure shows the energy levels labeled using the radial quantum number.

(4,0)=4s (4,1)=4p (4,2)=4d (4,3)=4f

(3,0)=3s (3,1)=3p (3,2)=3d

(2,0)=2s (2,1)=2p

(1,0)=1s

Energy levels of the hydrogen atom labeled with the principal quantum number in the ordinary way (n,l).

(3,0)=4s (2,1)=4p (1,2)=4d (0,3)=4f

(2,0)=3s (1,1)=3p (0,2)=3d

(1,0)=2s (0,1)=2p

(0,0)=1s

Energy levels of the hydrogen atom labeled with the radial quantum number (n',l).

The degeneracy of levels with the same principal quantum number does not have an analog in the spherical acoustic resonator, since the radial equation is different.

In the spherically symmetric case, the eigenvalues for different magnetic quantum numbers *m* are degenerate for any form of the radial equation. This is true for both the hydrogen atom and the spherical acoustic resonator. In general, the eigenvalues numbered by the quantum numbers (n,l) or by (n',l) are $(2l+1)$-fold degenerate. This degeneracy is lifted when the spherical symmetry is broken.

Now let's do some experiments that allow us to see many of these effects. First, we will identify the resonances by their angular dependence.

2.1 Measure resonances in the spherical resonator and determine their quantum numbers

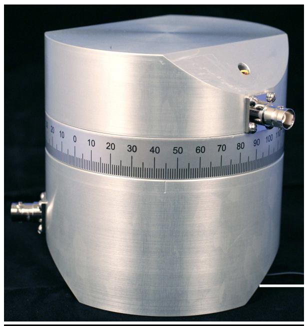
Objective: Determine the resonance frequencies for the spherical resonator and gather data to determine their angular-momentum quantum numbers.

Equipment Required:

TeachSpin Quantum Analog System: Controller, Hemispheres, Accessories
Sine wave generator capable of producing 1-50 kHz with a peak-to-peak voltage of 0.50 V
Two-Channel Oscilloscope

Setup:

Assemble two of the hemispheres so that the speaker is in the lower hemisphere and a microphone is in the upper hemisphere. (Looking carefully at the photo, you will see the speaker wire at the lower right.) Adjust the position of the upper hemisphere so that α =180° on the scale is at the reference mark. In this position, the speaker and the upper microphone are at opposite ends of a diameter. (The microphones will be one above the other.)



Atom Analog
Microphones are imbedded under BNC connectors.
Speaker is at lower right.

Attach a BNC splitter of "tee"to *SINE WAVE INPUT* on the Controller. Connect the output of your sine wave generator to one side of the splitter. Use a BNC cable to send the sound signal to Channel 1 of the oscilloscope. Plug the lead from the speaker on the lower hemisphere to *SPEAKER OUTPUT* on the Controller. The same sine wave now goes to both the speaker and Channel 1.

Use a BNC cable to connect the microphone output from the **upper** hemisphere to *MICROPHONE INPUT* on the Controller. Connect *AC MONITOR* on the Controller to Channel 2 of the oscilloscope to display the sound signal received by the microphone. Trigger the oscilloscope on Channel 1.

Use the *ATTENUATOR* dial on the Controller to keep the signal on Channel 2 from going off scale. Remember, with an attenuator, a higher reading on the dial gives a smaller signal. (Appendix 1 describes the function of each part of the Controller.)

Experiment:

Start at a low frequency and sweep the frequency up to about 8 kHz (8,000 Hz).

Write down all the resonance frequencies you observe. (If you listen carefully, you may actually hear some of them.)

Objective:

Observe, qualitatively, the way the amplitude of the resonance signal depends on the location of the microphone.

Experiment:

We will now gather data that will allow us to infer the angular quantum numbers of the resonances. Go to the second resonance, at about 3680 Hz. Fine-tune the frequency until it is as close as possible to the peak of the resonance. Shift the curves on the oscilloscope horizontally so that a maximum of the microphone signal (Channel 2) is located in the center of the image and marked by a vertical line. Now, watching the signal on the oscilloscope, slowly rotate the upper hemisphere, with respect to the lower one, from $\alpha = 180°$ to $\alpha = 0°$.

Questions:

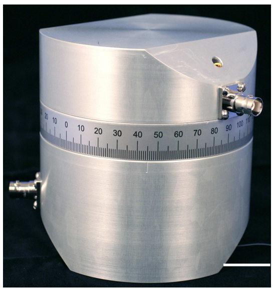
How did the amplitude change? Did the signal change its sign? Determine the angle where the amplitude is zero. At which angles is the signal maximal? Do both extrema have the same amplitude?

Note: Do not warm the aluminum parts too much by touching them with your hands. The speed of sound is temperature-dependent, and, in consequence, the resonance frequency would shift with temperature. While analyzing the angular dependence, the chosen generator frequency should remain on top of the resonance.

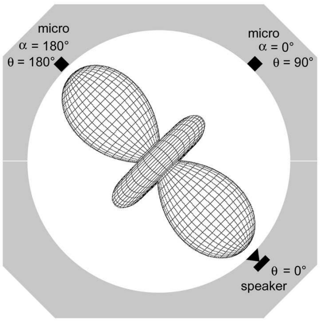
Analyze the data:

The angle α read on the scale is not a suitable angle for comparison with theory. Notice that the scale reading, α, running from 0 – 180°, tells you the rotation of the upper hemisphere about a vertical axis. The symmetry axis for this system, however, is determined by the speaker. The angle of interest, therefore, is measured using the speaker location as zero. To analyze the data, you must first use α to calculate the polar angle, θ. It is this angle which we use for polar coordinates. To clarify how this works try the following.

Assemble the sphere with the upper hemisphere set so that $\alpha = 180°$. Temporarily open the resonator and notice that at this setting the speaker and the upper microphone are 180 degrees apart in space; $\theta = 180°$. Reassemble the spheres, turn the upper sphere to $\alpha = 0°$, and open the resonator again. You will see that the spatial separation of the speaker and microphone, the polar angle of interest, is now $\theta = 90°$.



Atom Analog
Speaker is at lower right



Sample Sound Amplitude Pattern

Both the speaker and microphone are at an angle of 45° with respect to the horizontal plane between the hemispheres. By rotating the hemispheres with respect to each other, the angle θ can be changed from θ = 90° (at α = 0°) to θ = 180° (at α = 180°). Intermediate angles can be calculated using the formula

$$\theta = \arccos(\tfrac{1}{2}\cos\alpha - \tfrac{1}{2}) . \qquad (2.15)$$

You have measured the θ-dependence of the spherical harmonic function $Y_l^m(\theta,\varphi)$ with l = 2 and m = 0. Now we need to learn more about the spherical harmonics to compare the experiment with theory.

Derivation of equation 2.15

Assume that the speaker is located in the x-z-plane and the vertical axis is the z-axis. The position of the speaker in a sphere with unit-radius is given by the vector $\vec{s} = (\sqrt{\tfrac{1}{2}}, 0, -\sqrt{\tfrac{1}{2}})$.

We want to calculate the angle between speaker and microphone, which is the angle θ.

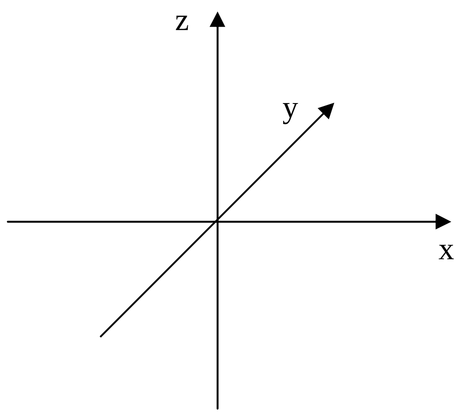
To calculate θ we use rotary matrices. In the first step, we rotate the vector $\vec{s}$ from the position of the speaker (vector $\vec{s}$) by 90 degrees around the y-axis arriving at $(\sqrt{\tfrac{1}{2}}, 0, \sqrt{\tfrac{1}{2}})$.

In the second step, we rotate by the angle α around the z axis. Lets call the resulting vector, $\vec{m}$, the position of the microphone.

From the scalar-product, $\vec{m}\cdot\vec{s} = |\vec{m}||\vec{s}|\cos\theta = \cos\theta$, we get the angle θ.

First rotation:

$$\begin{bmatrix} \cos 90° & 0 & -\sin 90° \\ 0 & 1 & 0 \\ \sin 90° & 0 & \cos 90° \end{bmatrix} \cdot \begin{bmatrix} \sqrt{\tfrac{1}{2}} \\ 0 \\ -\sqrt{\tfrac{1}{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{\tfrac{1}{2}} \\ 0 \\ \sqrt{\tfrac{1}{2}} \end{bmatrix}$$



Second rotation:

$$\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{\tfrac{1}{2}} \\ 0 \\ \sqrt{\tfrac{1}{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{\tfrac{1}{2}}\cos\alpha \\ \sqrt{\tfrac{1}{2}}\sin\alpha \\ \sqrt{\tfrac{1}{2}} \end{bmatrix}$$

scalar-product:

$$\vec{m}\cdot\vec{s} = \begin{bmatrix} \sqrt{\tfrac{1}{2}}\cos\alpha \\ \sqrt{\tfrac{1}{2}}\sin\alpha \\ \sqrt{\tfrac{1}{2}} \end{bmatrix} \cdot \begin{bmatrix} \sqrt{\tfrac{1}{2}} \\ 0 \\ -\sqrt{\tfrac{1}{2}} \end{bmatrix} = \frac{1}{2}\cos\alpha - \frac{1}{2}$$

with

$$\vec{m}\cdot\vec{s} = \cos\theta$$

we get the result:

$$\theta = \arccos\left(\frac{1}{2}\cos\alpha - \frac{1}{2}\right)$$

Spherical Harmonics and Legendre Polynomials:

The spherical harmonics $Y_l^m(\theta,\varphi)$ can be written as

$$Y_l^m(\theta,\varphi) \propto P_l^m(\cos\theta)\, e^{im\varphi} \qquad (2.16)$$

in terms of the associated Legendre polynomials P_l^m. For these experiments, we can restrict ourselves to the case m = 0, because our speaker creates waves with cylindrical symmetry about the speaker axis. For m = 0 the spherical harmonics do not have a φ-dependence and the wave function has the same amplitude for all azimuthal angles, φ. The dependence on the polar angle θ is given by the Legendre polynomials

$$Y_l^0(\theta,\varphi) \propto P_l^0(\cos\theta) \qquad (2.17)$$

The first nine Legendre polynomials are shown below:

$$P_0(\cos\theta) = 1$$
$$P_1(\cos\theta) = \cos\theta$$
$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$
$$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$
$$P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3)$$
$$P_5(\cos\theta) = \frac{1}{8}(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)$$
$$P_6(\cos\theta) = \frac{1}{16}(231\cos^6\theta - 315\cos^4\theta + 105\cos^2\theta - 5)$$
$$P_7(\cos\theta) = \frac{1}{16}(429\cos^7\theta - 693\cos^5\theta + 315\cos^3\theta - 35\cos\theta)$$
$$P_8(\cos\theta) = \frac{1}{128}(6435\cos^8\theta - 12012\cos^6\theta + 6930\cos^4\theta - 1260\cos^2\theta + 35)$$

In Fig. 2.1 and 2.2 the first six Legendre polynomials are plotted. The number of nodes in each Legendre polynomial is equal to the azimuthal quantum number *l*.

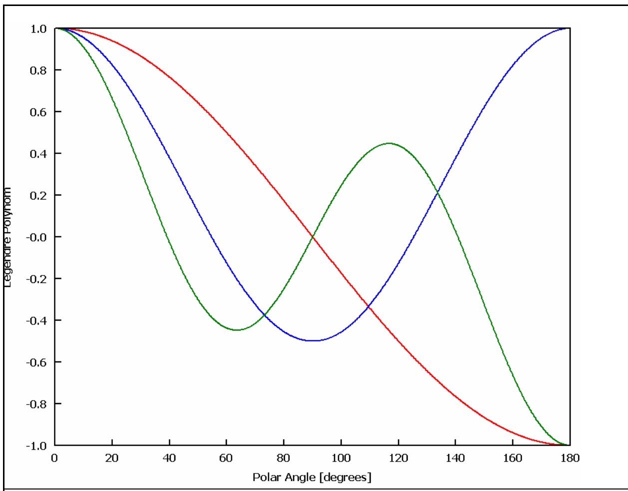


Fig. 2.1: Legendre Polynomials
$P_1(\cos\theta) = \cos\theta$ in red,
$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ in blue and
$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$ in green.

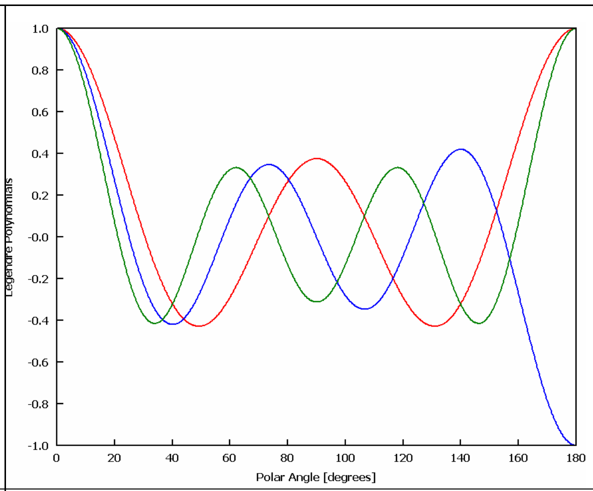


Fig. 2.2: Legendre Polynomials
$P_4(\cos\theta) = \frac{1}{8}(35\cos^4\theta - 30\cos^2\theta + 3)$ in red,
$P_5(\cos\theta) = \frac{1}{8}(63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)$ in blue
$P_6(\cos\theta) = \frac{1}{16}(231\cos^6\theta - 315\cos^4\theta + 105\cos^2\theta - 5)$
in green.

In the following table, the nodes of the Legendre polynomials are listed. **Be aware that these are the polar angles θ, and not the angles you read on the scale.**

P_0								
P_1	90°							
P_2	54.74°	125.26°						
P_3	39.23°	90°	140.77°					
P_4	30.56°	70.12°	109.88°	149.44°				
P_5	25.02°	57.42°	90°	122.58°	154.98°			
P_6	21.18°	48.61°	76.19°	103.81°	131.39°	158.82°		
P_7	18.36°	42.14°	66.06°	90°	113.94°	137.86°	161.64°	
P_8	16.20°	37.19°	58.30°	79.43°	100.57°	121.70°	142.81°	163.80°

Table 2.1: Nodes of the first eight Legendre polynomials, given in the polar angles θ.

Questions:

Now you can identify the angular quantum number *l* of the second resonance you have measured.

As you varied α from 180° to 0°, what range of θ did you cover?

How many nodes did you discover in the range you covered?

Based on your observations, to what *l* value does the resonance you examined correspond?

Does the θ angle measurement of the node you have measured agree with the angle predicted by the theory?

Do the relative magnitudes of the extrema fit to the theory?

Note about the magnetic quantum number:

The resonance that you have analyzed is $(2l+1)$-fold degenerate with respect to the magnetic quantum number *m*. However, in this experiment we observe almost exclusively the $m = 0$ state. The standing sound wave in the sphere is driven by the local speaker. The speaker defines the z-axis of the problem. It emits a wave traveling more or less back and forth along the z-axis and having cylindrical symmetry around that axis. This symmetry of the standing wave is described by the $m = 0$ state. States with other $m \neq 0$ describe waves that move on an orbit inside the sphere. These types of waves are much less effectively driven by our speaker located on the z-axis, since these states have nodes at $\theta = 0°$ and $\theta = 180°$.

Objective: We will trace out the angular dependence of the amplitude of the wave function.

Additional Apparatus: dc voltmeter

Setup:

As in the first part of this experiment, attach a BNC splitter to *SINE WAVE INPUT* on the Controller. Connect the output of your sine wave generator to one side of the splitter. Use a BNC cable to send the sound signal to Channel 1 of the oscilloscope. Plug the lead from the speaker on the lower hemisphere to *SPEAKER OUTPUT* on the Controller. The same sine wave now goes to both the speaker and Channel 1.

Use a BNC cable to connect the microphone output from the upper hemisphere to *MICROPHONE INPUT*. Connect *AC MONITOR* on the Controller to Channel 2 of the oscilloscope to display the sound signal received by the microphone. Trigger the oscilloscope on Channel 1.

This time put the upper hemisphere in the position α = 0° on the scale. In this position the microphone is directly above the speaker which means angle θ will be 90°.

To observe the amplitude of the sound signal at the microphone, connect a voltmeter to *DETECTOR-OUTPUT*. You should also observe the sound signal itself by connecting the *AC-MONITOR* on the Controller with Channel 2 of the oscilloscope. Trigger the oscilloscope to Channel 1.

Experiment:

For a couple of major resonances, measure the amplitude as function of the angle α. You can read the absolute value of the amplitude on the voltmeter and use the oscilloscope to determine the sign.

Record the nodes (angle at which the amplitude is zero) for the same resonances.

Analyze the data:

Plot your data as function of the polar angle θ and fit the data with the Legendre polynomial that is the best match. Do this for all the resonances you have measured.

Compare the nodes you have measured with the nodes of the corresponding Legendre polynomial given in table 2.1.

Note:

Some of the resonances are very close to each other so that the peaks are overlapping. This will result in a superposition of two wavefunctions with different quantum numbers. In this case, the angular dependence you have measured does not fit to a single Legendre polynomial. We will analyze these cases in more detail by taking spectra with the computer.

2.2 Measure spectra and wavefunctions in the spherical resonator with the computer

Objective: In this experiment, you will use a computer sound card both to generate the sound wave and to sweep its frequency. You will use the oscilloscope to observe the actual sine wave signals both going into the speaker and coming from the microphone. Simultaneously, you will use the computer to display a spectrum which shows the amplitude of the signal from the microphone as a function of the frequency of the sound.

Equipment Required:

TeachSpin Quantum Analog System: Controller, Hemispheres, Accessories
Two-Channel Oscilloscope
Two adapter cables (BNC - 3.5 mm plug)
Computer with sound card installed and Quantum Analogs “SpectrumSLC.exe” running

WARNING: The BNC-to-3.5-mm adapter cables are provided as a convenient way to couple signals between the Controller and sound card. Unfortunately, they could also provide a way for excessive external voltage sources to damage a sound card. Most sound cards are somewhat protected against excessive inputs, but ***it is the user's responsibility to ensure that adapter cable voltages are kept BELOW 5 Volts peak-to-peak***.
The maximum peak-to-peak value for optimum performance of the Quantum Analogs system depends on your sound card and can vary from 500 mV to 2 V.

Setup:

Now, using connectors on the Controller, you will send the sound card signal to both the speaker and Channel 1 of the oscilloscope, and the microphone signal to both the microphone input of the computer and to Channel 2 of the oscilloscope.

First, make sure that the *ATTENUATOR* knob on the Controller is set at 10 (out of 10) turns.

Let’s start with the sound signal. Attach a BNC splitter or “tee” to *SINE WAVE INPUT* on the Controller. Using the adapter cable, connect the output of the sound card to one arm of the splitter. With a BNC cable, convey the sound card signal from the splitter to Channel 1 of your oscilloscope. Plug the lead from the speaker on the lower hemisphere to *SPEAKER OUTPUT* on the Controller. The sound card signal is now going to both the speaker and Channel 1.

The microphone signal will also be sent two different places. Connect the microphone on the upper hemisphere to *MICROPHONE INPUT* on the Controller. Put a BNC splitter on the Controller connector labeled *AC-MONITOR*. From the splitter, use an adapter cable to send the microphone signal to the microphone input on the computer sound card. Use a BNC cable to send the same signal to Channel 2 of the oscilloscope to show the actual signal coming from the microphone.

The computer will plot the instantaneous frequency generated by the sound card on the x-axis and the amplitude of the microphone input signal on the y-axis.

The next job is to adjust the magnitude of both the speaker and microphone signals so that you will have maximum signal while keeping the microphone input to the computer from saturating. Peak-to-peak signals to the microphone input can range from 0.50 to 2.0 volts depending upon your sound card.

Once the program, SpectrumSLC.exe., is running, you can configure the computer. Go to the menu at the top of the screen and choose Configure > Input Channel/Volume At this point, choose *Line In*, if it is available; otherwise choose *Microphone*. On this screen, set the microphone volume slider to the middle of its range.

To set the speaker volume, use the *Amplitude Output Signal* on the lower left of the computer screen. That slider should also be set to middle range.

The microphone signal coming from the apparatus first passes through a built-in amplifier, and then through the *ATTENUATOR*, before reaching the *AC-MONITOR* connector. The ten-turn knob on the attenuator *decreases* the incoming signal by a factor ranging from zero to 100. For example, a setting of 9.0 turns (out of the 10 turns possible) stands for an attenuation of 9/10 or 90% attenuation of the signal. (A higher setting means a smaller signal.)

After taking an initial wide range spectrum, choose a section that includes the highest peak and a smaller one next to it. Readjust the scan to cover just this portion. Using the option that allows you to keep successive spectra visible, take Spectrum 1, 2, 3, etc. with the attenuator knob set at 9.9, 9.8, 9.7, . . . turns (out of ten). The nesting heights of the peaks will tell you whether or not the system is behaving in a linear fashion. Continue to go lower on the 10-turn dial setting until the computer program flashes 'saturation'. You will also have visual evidence of saturation – a flat section on the tallest peak or a smaller "nesting" spacing. (See Appendix 2 or 3 for details.)

Once you have reached saturation, drop back into the linear range. Now you can operate with confidence that the signals you see really are proportional to the amplitude of the sound wave you are studying.

Experiment:

Set the hemispheres so that the scale angle $\alpha = 180°$.

Start the program SpectrumSLC.exe and measure an overview spectrum. You can use coarse steps such as 10 Hz and a short time per step such as 50 ms.

Change the angle between the upper and the lower hemisphere several times and observe the how the spectrum changes. Be sure to look at the spectrum for $\alpha = 0°$.

Question: What changes do you notice?

Experiment:

Go back to $\alpha = 0°$ and look in more detail at the peak near 5000 Hz. Actually, there are two peaks close to each other. Take a spectrum that measures slow enough and with sufficiently small steps to show the details of these two peaks. Also, take spectra for this range at $\alpha = 20°$ and $\alpha = 40°$.

Question: What do you notice?

Objective: Create polar plots for a series of resonances and use the plots to identify the angular momentum number and spherical harmonic function of each resonance.

Experiment:

Now we will measure the wavefunctions of the different resonances and visualize them by a polar plot of the amplitude $A(\theta)$. The computer calculates the polar angle θ from the angle α and it plots the absolute value of the amplitude as function of θ in a polar plot. This diagram makes it easy to identify the angular quantum number and the spherical harmonic function.

Take a spectrum with α = 180° from 2000 Hz to 7000 Hz sufficiently slowly. If you click with the left mouse button on a peak, the output frequency is adjusted to the value at which you clicked. Look at the oscilloscope and convince yourself that you are at a resonance. In the computer menu, go to "Windows" > "Measure Wave Function".

Adjust the hemispheres to α = 0°, and measure the amplitude in steps of 10°. The program converts the angle α automatically to the polar angle θ and plots the absolute of the amplitude in a polar plot. Use the function "complete by symmetry" to complete the figure.

Create polar-plots for the prominent peaks and identify the quantum numbers.

Analyze data:

Compare the polar plots you have generated with polar-plots of the Legendre polynomials. Some of them are given below, the others you can visualize with the program PlotYlm.exe.

In case of overlapping peaks, you will find distorted figures, since there are contributions to the wave functions from two different eigenstates with different quantum numbers and symmetries.

Fig. 2.3: Plots of the spherical harmonics:

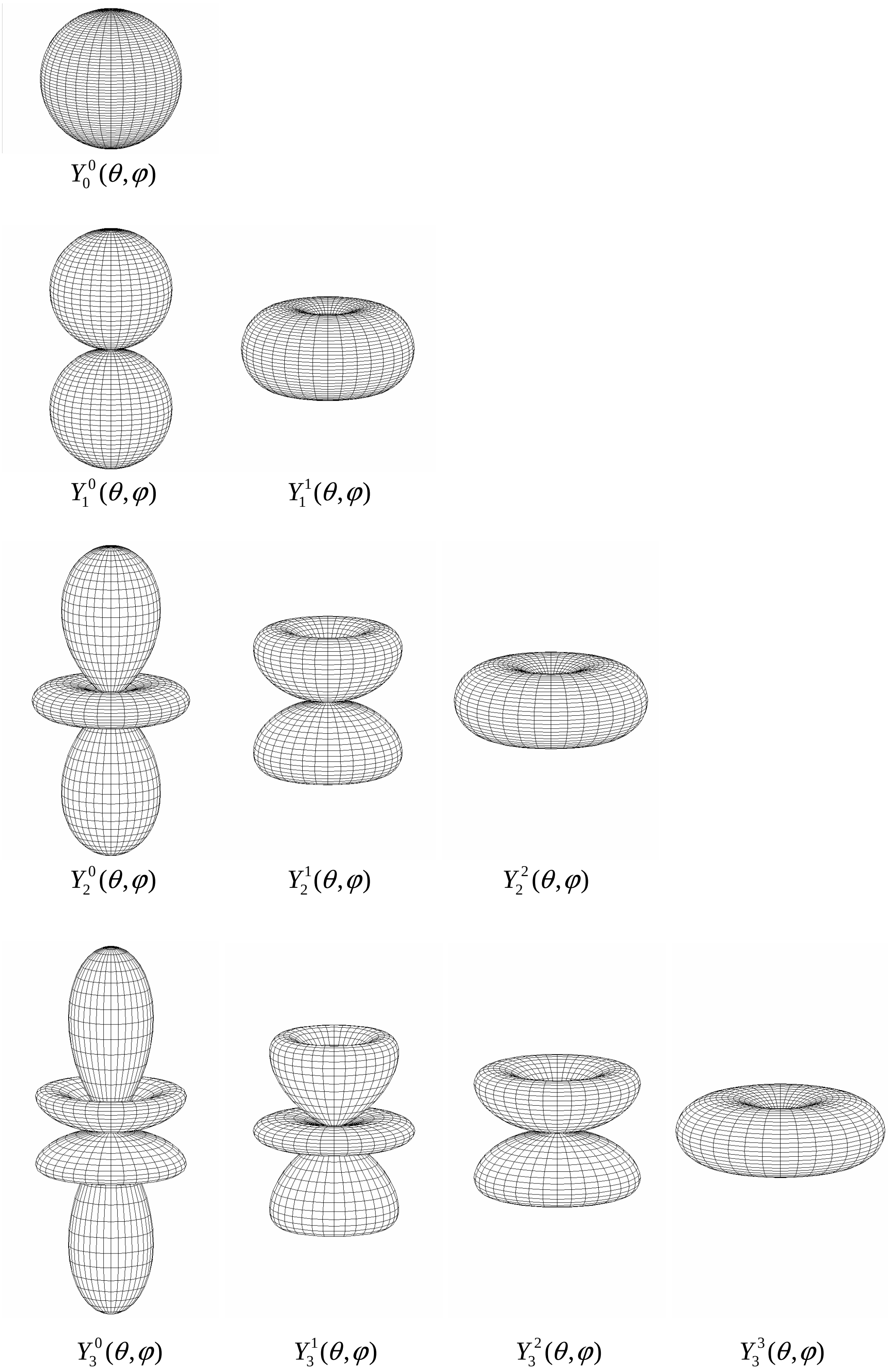
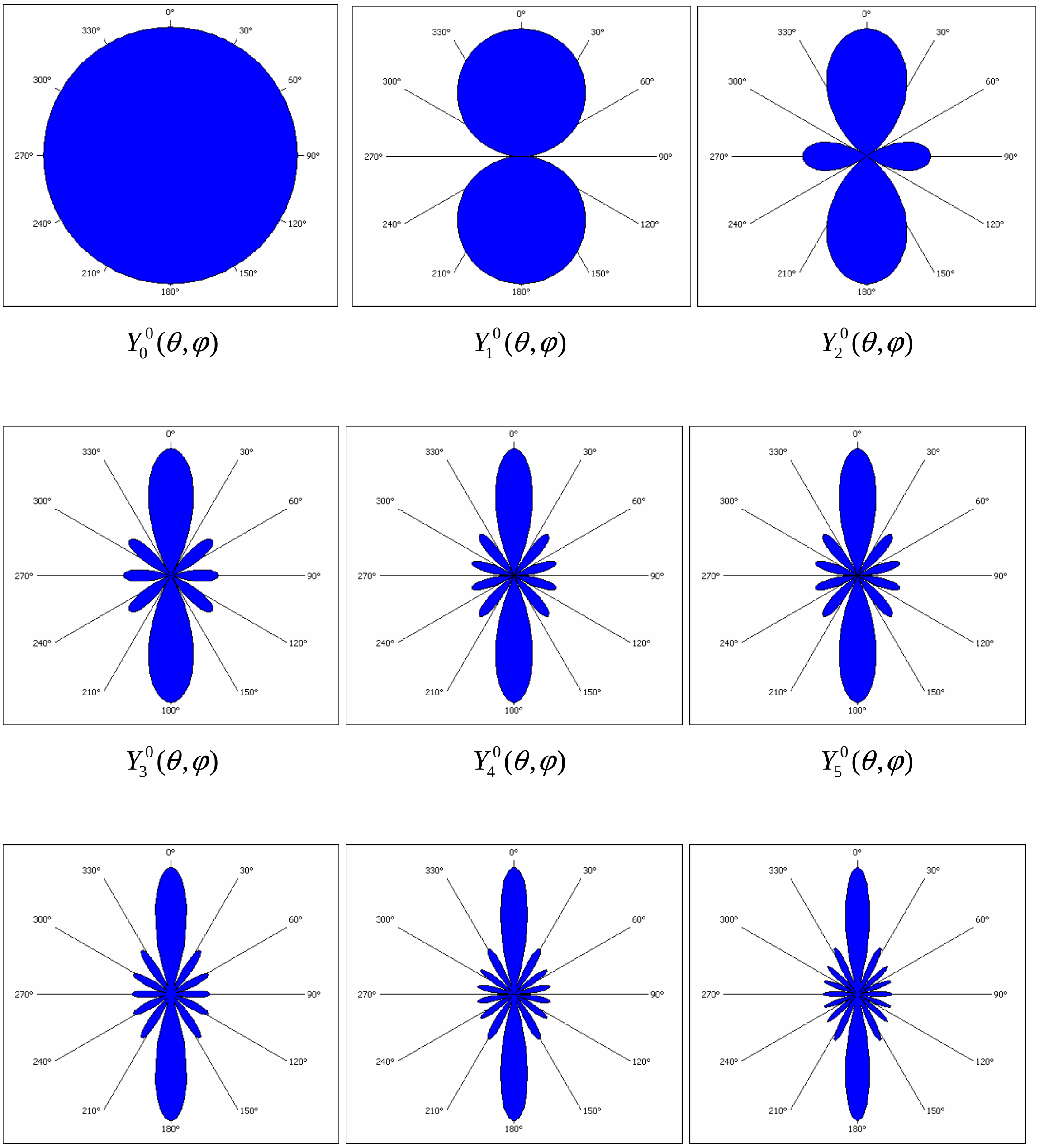


Fig. 2.4: Cut through the spherical harmonics with magnetic quantum number *m* = 0.



$Y_0^0(\theta,\varphi)$ $Y_1^0(\theta,\varphi)$ $Y_2^0(\theta,\varphi)$

$Y_3^0(\theta,\varphi)$ $Y_4^0(\theta,\varphi)$ $Y_5^0(\theta,\varphi)$

$Y_6^0(\theta,\varphi)$ $Y_7^0(\theta,\varphi)$ $Y_8^0(\theta,\varphi)$